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Euclidean limit of L^2-spectral properties of the Pauli Hamiltonians on constant curvature Riemann surfaces

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Abstract

We realize the Pauli Hamiltonians $\mathbb{P}_\kappa^\nu$ (with constant magnetic field $\nu > 0$) on a simply connected Riemann surface M_κ of constant scalar curvature $\kappa \in \mathbb{R}$ as second-order differential operators acting on differential 1-forms of M_κ. We also study the asymptotic behaviour of some aspects of their L^2-spectral properties when the Euclidean limit is taken. More exactly, we show that the L^2-eigenprojector kernels on the plane $\mathbb{R}^2 = \mathbb{C}$ (i.e., $\kappa = 0$) corresponding to the Landau levels $8\nu l$; $l = 0, 1, \ldots,$ can be recovered from the L^2-eigenprojector kernels of $\mathbb{P}_\kappa^\nu$ of the curved Riemann surfaces M_κ, $\kappa \neq 0$, in the limit $\kappa \longmapsto 0$.

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1. Introduction

A single non-relativistic spinless particle constrained to move on a two-dimensional analytic surface M_κ of constant scalar curvature κ, in the presence of a uniform external constant magnetic field of magnitude $\nu > 0$ directed orthogonally, is described by the Landau Hamiltonian $\mathbb{L}_\kappa^\nu$ [2, 5, 6] given explicitly in z-complex notation by

$$\mathbb{L}_\kappa^\nu = -(1+\kappa|z|^2)\left\{(1+\kappa|z|^2)\frac{\partial^2}{\partial z\partial\bar{z}} + \nu\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}\right)\right\} + \nu^2|z|^2. \quad (1)$$

The above Landau Hamiltonians $\mathbb{L}_\kappa^\nu$ (or Maass Laplacians) have been extensively studied in the physics and mathematics literature by many authors using different approaches; see, for example, [2, 6, 9, 10]. They can be realized as Schrödinger operators by considering

$$\mathbb{L}_\kappa^\nu = (d + \mathrm{i}\nu\,\mathbf{ext}(\theta_\kappa))^*(d + \mathrm{i}\nu\,\mathbf{ext}(\theta_\kappa)) \quad (2)$$

acting on functions, with $[\mathbf{ext}(\theta_\kappa)f](z) := f(z)\theta_\kappa(z)$ and where the differential 1-form θ_κ is the gauge vector potential given explicitly by

$$\theta_\kappa(z) = \frac{\mathrm{i}(\bar{z}\,\mathrm{d}z - z\,\mathrm{d}\bar{z})}{1+\kappa|z|^2}.$$

Then, it is clear that the usual Landau Hamiltonian $\mathbb{L}^\nu = \mathbb{L}^\nu_0$ on $\mathbb{R}^2 = \mathbb{C}$ given by

$$\mathbb{L}^\nu = -\left\{ \frac{\partial^2}{\partial z \partial \bar{z}} + \nu \left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}} \right) - \nu^2 |z|^2 \right\}, \tag{3}$$

can be recovered formally as a limit of the unbounded operators $\mathbb{L}^\nu_\kappa$ when $\kappa \to 0$.

Thus the problem of connecting the spectral properties of the Landau Hamiltonian $\mathbb{L}^\nu$ on $\mathbb{C}$ as a limit of those on the curved spaces follows. This was first analysed by Comtet in [3]. He had shown in particular that on the Poincaré upper half plane $\mathcal{P} = \{(x, y) \in \mathbb{R}^2; y > 0\}$ endowed with the scaled hyperbolic metric whose negative constant scalar curvature is $-1/\rho^2$, the spectrum of the associated Landau Hamiltonian given by

$$\mathbb{L}^\nu(\rho) = -\left\{ \frac{y^2}{\rho^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - 2\mathrm{i}\nu y \frac{\partial}{\partial x} - \nu^2 \rho^2 \right\}$$

gives rise, when $\rho \to \infty$, to the well-known Landau energy levels $\nu(2m+1)$, $m = 0, 1, \ldots,$ that constitute the L^2-point spectrum of the usual Landau Hamiltonian $\mathbb{L}^\nu$ on $L^2(\mathbb{R}^2; \mathrm{d}x\,\mathrm{d}y)$. Since then many authors have been interested. For the hyperbolic disc D_ρ of radius $\rho > 0$ the situation is pretty similar since the above Landau Hamiltonian $\mathbb{L}^\nu(\rho)$ is unitary equivalent to $\mathbb{L}^\nu_\kappa$, (1), on D_ρ with $\kappa = -1/\rho^2$, via the Cayley transform $w \mapsto z = \rho(w - i)/(w + i)$. The generalization to higher dimensions, i.e., the complex Bergman ball $\mathbb{B}^n_\rho$ is presented in [7]. For the compact partner of D_ρ, i.e., the sphere $S^2_\rho \subset \mathbb{R}^3$ whose positive constant scalar curvature is $+1/\rho^2$, one can refer to [8].

In this paper, interested by a similar problem, we study the Euclidean limit of the L^2-spectral properties of the Pauli Hamiltonian $\mathbb{P}^\nu_\kappa$ on the constant curvature Riemann surfaces M_κ (see (7) below). For the flat case ($\kappa = 0$), the Pauli Hamiltonian $\mathbb{P}^\nu = \mathbb{P}^\nu_0$ that describes a non-relativistic spin particle, acting on the two-component spinor $\binom{\varphi}{\chi}$, is known to be given explicitly by

$$\mathbb{P}^\nu = \begin{pmatrix} \mathbb{L}^\nu & 0 \\ 0 & \mathbb{L}^\nu \end{pmatrix} - \nu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

Therefore, the study of the L^2-spectral properties of $\mathbb{P}^\nu$ reduces further to the usual scalar Landau Hamiltonian $\mathbb{L}^\nu$ [9, 1, 4]. Such a Pauli Hamiltonian on $\mathbb{R}^2 = \mathbb{C}$ can also be realized, such as the Landau Hamiltonian (2), as a second-order differential operator acting on differential 1-forms $\omega = \varphi\,\mathrm{d}z + \chi\,\mathrm{d}\bar{z}$ of $\mathbb{C}$ through

$$\mathbb{P}^\nu = (d + \mathrm{i}\nu\,\mathbf{ext}(\theta))^*(d + \mathrm{i}\nu\,\mathbf{ext}(\theta)) + (d + \mathrm{i}\nu\,\mathbf{ext}(\theta))(d + \mathrm{i}\nu\,\mathbf{ext}(\theta))^*, \tag{5}$$

with $\theta = \mathrm{i}(\bar{z}\,\mathrm{d}z - z\,\mathrm{d}\bar{z})$ and $\mathbf{ext}(\theta)\omega = \theta \wedge \omega$. See also [14] for details on a similar realization. Let us mention here that the Pauli Hamiltonian $\mathbb{P}_\nu$ derived in (5) and given explicitly in (4) is equivalent to the standard magnetic Schrödinger operator given by

$$(-\mathrm{i}\nabla - \overrightarrow{A})^2 + \overrightarrow{\beta} \cdot \overrightarrow{B},$$

where $\overrightarrow{A} = 2\nu(y, -x, 0) = \nu\theta$ is the vector potential, $\overrightarrow{B} = \nabla \times \overrightarrow{A} = (0, 0, -4\nu)$ is the associated magnetic field and $\overrightarrow{\beta} = (0, 0, \pm 1)$ is the spin direction.

Then, to extend the notion of the Pauli Hamiltonian to any Riemann surface M_κ, one can consider again (5) with θ_κ instead of θ. This will be considered as a geometrical realization of the Pauli Hamiltonians on constant curvature Riemann surfaces. The concrete study of the L^2-spectral properties such as the L^2-eigenvalues and the explicit expressions of the associated L^2-eigenforms as well as their asymptotics, when $\kappa \to 0$, are so obtained by reducing further to a system of two Landau Hamiltonians (lemma 2). This is possible thanks to the explicit expression in z-complex notation of $\mathbb{P}^\nu_\kappa$ established in theorem 1.

Also, we have to show that for every fixed $(z, w) \in \mathbb{C} \times \mathbb{C}$ the L^2-eigenprojector kernel $\mathcal{K}^{\nu}_{\kappa;l}(z, w)$ (reproducing kernel) of the Pauli Hamiltonian $\mathbb{P}^{\nu}_{\kappa}$ on M_κ, for κ small enough, converges pointwise to the L^2-eigenprojector kernel $\mathcal{K}^{\nu}_{l}(z, w)$ of the Pauli Hamiltonian $\mathbb{P}^{\nu}$ on $\mathbb{C}$ (see theorem 4). This can give, somehow, a justification of the formal limit of $\mathbb{P}^{\nu}_{\kappa}$ (or resp. $\mathbb{L}^{\nu}_{\kappa}$) to $\mathbb{P}^{\nu}$ (resp. $\mathbb{L}^{\nu}$) when $\kappa \to 0$. We will carry out our study for the three models of simply connected Riemann surfaces with constant scalar curvature in an unified manner as in [5] or [12].

Let us note here that one cannot use perturbative theory theorems to obtain the above results as done in [13]; the situation here is quite different.

The paper is structured as follows. In section 2, we fix notation, realize the Pauli Hamiltonians on M_κ, as second-order differential operators on differential 1-forms, and give their explicit expressions in z-complex notation (theorem 1) as well as their invariance property. In section 3, we provide concrete description of their L^2-eigenforms, while in section 4, we give the explicit closed formulae for the L^2-eigenprojector kernels of their corresponding L^2-eigenspaces (see theorem 3). The asymptotic of such L^2-eigenprojector kernels, when $\kappa \to 0$ is proved in section 5. In section 6, we present some related remarks. We conclude with an appendix in which we give the proofs of theorem 1 and proposition 4.

2. Pauli Hamiltonians on constant curvature Riemann surfaces

Let M_κ be a simply connected Riemann surface with constant scalar curvature $\kappa(\kappa = \varepsilon/\rho^2$ with $\varepsilon = \mp 1$ fixed and $\rho \in]0, +\infty]$ with the convention that $\kappa = 0$ whenever $\rho = +\infty)$. Then, it is known (Riemann uniformization theorem) that M_κ can be realized as the disc, the plane or the sphere in $\mathbb{R}^3$. That is

$$M_\kappa = \begin{cases} D_\rho = \{z \in \mathbb{C};\, |z| < \rho\} & \text{for} \quad \kappa = -\frac{1}{\rho^2} < 0 \\ \mathbb{C} & \text{for} \quad \kappa = 0 \\ S^2_\rho = \mathbb{C} \cup \{\infty\} & \text{for} \quad \kappa = +\frac{1}{\rho^2} > 0. \end{cases}$$

We equip M_κ with the Hermitian metric $\mathrm{d}s^2_\kappa$ given by

$$\mathrm{d}s^2_\kappa := \frac{4}{(1 + \kappa|z|^2)^2}\, \mathrm{d}z \otimes \mathrm{d}\bar{z},$$

and let θ_κ and $\mathrm{d}\mu_\kappa$ be the associated real differential 1-form (a gauge vector potential) and the volume measure on M_κ given respectively by

$$\theta_\kappa(z) = \frac{\mathrm{i}(\bar{z}\,\mathrm{d}z - z\,\mathrm{d}\bar{z})}{1 + \kappa|z|^2} \qquad \text{and} \qquad \mathrm{d}\mu_\kappa(z) = \frac{4\,\mathrm{d}m(z)}{(1 + \kappa|z|^2)^2}, \tag{6}$$

where $\mathrm{d}m$ denotes the usual Lebesgue measure on M_κ.

Next, for $\nu \in \mathbb{R}$, we denote by $\nabla^{\nu}_{\theta_\kappa}$ the first-order differential operator acting on differential p-forms of M_κ by $\nabla^{\nu}_{\theta_\kappa} := d + \mathrm{i}\nu(\mathbf{ext}\,\theta_\kappa)$. Here d is the usual exterior derivative and $\mathbf{ext}\,\theta_\kappa$ is the operator of exterior left multiplication by θ_κ, i.e., $\mathbf{ext}(\theta_\kappa)\omega = \theta_\kappa \wedge \omega$ for all differential p-form ω. By $\left(\nabla^{\nu}_{\theta_\kappa}\right)^*$ let us denote the formal adjoint of $\nabla^{\nu}_{\theta_\kappa}$ with respect to the Hermitian scalar product induced by the Hermitian metric $\mathrm{d}s^2_\kappa$.

Definition. *We call the Pauli Hamiltonian associated with the vector potential θ_κ on M_κ the following second-order differential operator $\mathbb{P}^{\nu}_{\kappa}$ acting on differential 1-forms of M_κ by*

$$\mathbb{P}^{\nu}_{\kappa} := \left(\nabla^{\nu}_{\theta_\kappa}\right)^* \nabla^{\nu}_{\theta_\kappa} + \nabla^{\nu}_{\theta_\kappa}\left(\nabla^{\nu}_{\theta_\kappa}\right)^*. \tag{7}$$

The above realization (7) allows one to show easily that such Pauli Hamiltonians $\mathbb{P}_\kappa^\nu$ *are invariant by the action of the group of motions*

$$G_\kappa := \left\{ g = \begin{pmatrix} a & b \\ -\kappa\bar{b} & \bar{a} \end{pmatrix} = g(\kappa) \in M_{2,2}(\mathbb{C});\ |a|^2 + \kappa|b|^2 = 1 \text{ and } \lim_{\kappa\to 0} g(\kappa) \text{ exists} \right\}$$

acting on M_κ *via the transitive action defined by* $g \cdot z = (az+b)(-\kappa\bar{b}z+\bar{a})^{-1}$ *for* $g \in G_\kappa$. *More precisely, let* T_κ^ν *be the unitary projective representation of* G_κ *on the Hilbert space* $\mathcal{H}_\kappa := L^2(M_\kappa; \mathrm{d}m)\,\mathrm{d}z \oplus L^2(M_\kappa; \mathrm{d}m)\,\mathrm{d}\bar{z}$ *defined by*

$$\left[T_\kappa^\nu(g)\omega\right](z) := j_\kappa^\nu(g,z) g^*(\omega)(z), \qquad \text{with} \quad j_\kappa^\nu(g,z) := \left(\frac{1+\kappa\bar{z}g^{-1}\cdot 0}{1+\kappa z\overline{g^{-1}\cdot 0}} \right)^{\frac{\nu}{\kappa}},$$

where $g^*\omega$ *is the pull back of the differential form* ω *by the biholomorphic mapping* $g : z \mapsto g\cdot z$ *for fixed* $g \in G_\kappa$, *then we have*

Proposition 1 [7] (Invariance property)**.** *The Pauli Hamiltonians* $\mathbb{P}_\kappa^\nu$ *are invariant by the projective representation* T_κ^ν *of the group* G_κ *on* M_κ. *That is, for all* $g \in G_\kappa, \omega \in \mathcal{H}_\kappa$ *and* $z \in M_\kappa$, *we have*

$$T_\kappa^\nu(g)\left[\mathbb{P}_\kappa^\nu(\omega)\right](z) = \mathbb{P}_\kappa^\nu\left[T_\kappa^\nu(g)\omega\right](z).$$

Proof. First let us note that if the unitary transformation T_κ^ν commutes with $\nabla_{\theta_\kappa}^\nu$ then it commutes also with $\left(\nabla_{\theta_\kappa}^\nu\right)^*$ and so the assertion of the proposition follows. The identity $T_\kappa^\nu(g)\nabla_{\theta_\kappa}^\nu = \nabla_{\theta_\kappa}^\nu T_\kappa^\nu(g)$, for all $g \in G_\kappa$, holds by the use of the known facts $\mathrm{d}g^*\omega = g^*\mathrm{d}\omega$ and $g^*(\theta \wedge \omega) = g^*\theta \wedge g^*\omega$ combined with the following lemma:

Lemma 1. *For* $g \in G_\kappa$ *and* $z \in M_\kappa$, *we have*

$$\left[T_\kappa^\nu(g)(\theta_\kappa)\right](z) = j_\kappa^\nu(g,z)\theta_\kappa(z) - \frac{\mathrm{i}}{\kappa} d\left[j_\kappa^\nu(g,z)\right]. \qquad \square$$

Now, to describe the concrete spectral properties of such Pauli Hamiltonians $\mathbb{P}_\kappa^\nu$ we need first to have their explicit expressions. Precisely, we have

Theorem 1. *The Pauli Hamiltonian* $\mathbb{P}_\kappa^\nu$ *as defined in (7) acts on smooth differential 1-forms* $\omega = f\,\mathrm{d}z + g\,\mathrm{d}\bar{z}$, *identified to* $\binom{f}{g}$, *through the following explicit* 2×2 *matrix differential operator given by*

$$\mathbb{P}_\kappa^\nu = \begin{pmatrix} \mathbb{L}_\kappa^{\nu,\nu-2\kappa} & 0 \\ 0 & \mathbb{L}_\kappa^{\nu+2\kappa,\nu} \end{pmatrix} - \nu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{8}$$

where $\mathbb{L}_\kappa^{\alpha,\beta}$, $\alpha, \beta \in \mathbb{R}$, *is the second-order differential operator acting on scalar functions on* M_κ *by*

$$\mathbb{L}_\kappa^{\alpha,\beta} = -(1+\kappa|z|^2)\left\{(1+\kappa|z|^2)\frac{\partial^2}{\partial z\partial\bar{z}} + \alpha z\frac{\partial}{\partial z} - \beta\bar{z}\frac{\partial}{\partial\bar{z}}\right\} + \alpha\beta|z|^2.$$

The proof of theorem 1 is technical and will be given in appendix A.

Remark 1. Using the obtained explicit formula of the Pauli Hamiltonian, it can be shown that $\mathbb{P}_\kappa^\nu$ is an elliptic operator, densely defined on the Hilbert space $\mathcal{H}_\kappa = L^2(M_\kappa; \mathrm{d}m)\,\mathrm{d}z \oplus L^2(M_\kappa; \mathrm{d}m)\,\mathrm{d}\bar{z}$ and admits unique self-adjoint realization on $\mathcal{H}_\kappa$ that we denote also by $\mathbb{P}_\kappa^\nu$.

3. Concrete description of the L^2-eigenforms of $\mathbb{P}_\kappa^\nu$

In this section we are concerned with the concrete description of the L^2-eigenforms $\omega \in \mathcal{H}_\kappa$ of the Pauli Hamiltonian $\mathbb{P}_\kappa^\nu$. To do this we begin first by a brief review of some well-established L^2-spectral properties of the Landau Hamiltonians $\mathbb{L}_\kappa^{\tilde\nu}$, $\tilde\nu > 0$, on $L^2(M_\kappa; \mathrm{d}\mu_\kappa) = \left\{f : M_\kappa \longrightarrow \mathbb{C}; \int_{M_\kappa} |f(z)|^2 \,\mathrm{d}\mu_\kappa < \infty\right\}$.

3.1. L^2-spectral properties of the Landau Hamiltonian $\mathbb{L}_\kappa^{\tilde\nu}$

For $\tilde\nu > 0$ let $\mathbb{L}_\kappa^{\tilde\nu}$ be the usual Landau Hamiltonian on the Hilbert space $L^2(M_\kappa; \mathrm{d}\mu_\kappa)$ defined by

$$\mathbb{L}_\kappa^{\tilde\nu} = -(1+\kappa|z|^2)\left\{(1+\kappa|z|^2)\frac{\partial^2}{\partial z\partial\bar z} + \tilde\nu\left(z\frac{\partial}{\partial z} - \bar z\frac{\partial}{\partial \bar z}\right)\right\} + \tilde\nu^2|z|^2 =: \mathbb{L}_\kappa^{\tilde\nu,\tilde\nu} \tag{9}$$

that we can realize it also as $\mathbb{L}_\kappa^{\tilde\nu} = \left(\nabla_{\theta_\kappa}^{\tilde\nu}\right)^*\nabla_{\theta_\kappa}^{\tilde\nu}$ acting on scalar functions on M_κ. Thus, let us consider the following eigenvalue problem:

$$\mathbb{L}_\kappa^{\tilde\nu}\phi = \mu\phi, \qquad \phi \in L^2(M_\kappa; \mathrm{d}\mu_\kappa), \qquad \mu \in \mathbb{C}. \tag{10}$$

Then, we have

Proposition 2. *(i) The discrete part* $\mathrm{Spec}_d\left(\mathbb{L}_\kappa^{\tilde\nu}\right)$ *of the spectrum of the Landau Hamiltonian* $\mathbb{L}_\kappa^{\tilde\nu}$ *on* $L^2(M_\kappa; \mathrm{d}\mu_\kappa)$ *is given by*

$$\mathrm{Spec}_d\left(\mathbb{L}_\kappa^{\tilde\nu}\right) = \left\{\mu_\kappa(m) := \tilde\nu(2m+1) + \kappa m(m+1), \quad m \in \mathbb{Z}^+, \quad 0 \leqslant m < \frac{2\tilde\nu + \kappa}{|\kappa| - \kappa}\right\}$$

with the conditions $2\tilde\nu + \kappa > 0$ *for* $\kappa \leqslant 0$ *and* $2\tilde\nu/\kappa \in \mathbb{Z}^+$ *for* $\kappa > 0$.

(ii) A smooth function $\phi \in L^2(M_\kappa; \mathrm{d}\mu_\kappa)$ *is a solution of (10) if and only if* $\mu = \mu_\kappa(m)$. *In this case,* ϕ *is expanded explicitly in terms of the Gauss hypergeometric function* ${}_2F_1(a, b; c; x)$ *as follows*

$$\phi(z) = \sum_{p=0}^{+\infty}\sum_{q=0}^{m} a_{pq}^m \phi_{\kappa,m}^{\tilde\nu,pq}(z), \tag{11}$$

where

$$\phi_{\kappa,m}^{\tilde\nu,pq}(z) := (1+\kappa|z|^2)^{-\frac{\tilde\nu}{\kappa}-m}{}_2F_1\left(q-m,\, p-m-\frac{2\tilde\nu}{\kappa};\, p+q+1;\, -\kappa|z|^2\right) z^p\bar z^q \tag{12}$$

for $p, q \in \mathbb{Z}^+$ *such that* $pq = 0$. *The complex numbers* a_{pq}^m *satisfies the following growth condition:*

$$\sum_{p=0}^{+\infty}\frac{m!(p!)^2}{2(p+m)!}\cdot\left(\frac{|\kappa|^{-p}\Gamma\left(\frac{2\tilde\nu}{\kappa} - m\right)}{(2\tilde\nu + \kappa(2m+1))\Gamma\left(\frac{2\tilde\nu}{\kappa} + p - m\right)}\right)\left|a_{p0}^m\right|^2 < +\infty. \tag{13}$$

Proof. The result in (i) is well known in the literature of mathematics and physics as Landau energy levels. See for instance [2, 3, 6]. For the first part of (ii) (i.e., (11), (12)); the reader can refer to [3] and [11] for example. The growth condition (13) for the coefficients a_{pq}^m is given in [7]. □

Remark 2. The continuous part $\mathrm{Spec}_c\left(\mathbb{L}_\kappa^{\tilde\nu}\right)$ of the spectrum of the Landau Hamiltonian $\mathbb{L}_\kappa^{\tilde\nu}$ on $L^2(M_\kappa; \mathrm{d}\mu_\kappa)$ is empty for $\kappa \geqslant 0$. For the disc D_ρ of radius ρ (i.e., $\kappa = \frac{-1}{\rho^2} < 0$), it is given by

$$\mathrm{Spec}_c\left(\mathbb{L}_\kappa^{\tilde\nu}\right) = \left[-\frac{\kappa}{4} - \frac{\nu^2}{\kappa}, +\infty\right[= \left[\frac{1}{4\rho^2} + \nu^2\rho^2, +\infty\right[. \tag{14}$$

3.2. L^2-eigenforms of the Pauli Hamiltonian $\mathbb{P}_\kappa^\nu$

We fix $\nu > 0$ and consider the following eigenvalue problem for the Pauli Hamiltonian $\mathbb{P}_\kappa^\nu$ on $\mathcal{H}_\kappa = L^2(M_\kappa; \mathrm{d}m)\,\mathrm{d}z \oplus L^2(M_\kappa; \mathrm{d}m)\,\mathrm{d}\bar{z}$:

$$\mathbb{P}_\kappa^\nu \omega = \lambda\omega, \qquad \omega = f\,\mathrm{d}z + g\,\mathrm{d}\bar{z} \in \mathcal{H}_\kappa, \qquad \lambda \in \mathbb{C}. \tag{15}$$

According to the explicit expression of $\mathbb{P}_\kappa^\nu$ given in theorem 1, the above equation (15) reduces further to that of the scalar Landau Hamiltonian $\mathbb{L}_\kappa^{\tilde{\nu}}$. Namely, we have

Lemma 2. *Let*

$$\mathbf{S}_\kappa = \begin{pmatrix} 1+\kappa|z|^2 & 0 \\ 0 & 1+\kappa|z|^2 \end{pmatrix}.$$

Then

$$\mathbb{P}_\kappa^\nu = \mathbf{S}_\kappa^{-1}\left\{\begin{pmatrix} \mathbb{L}_\kappa^{\nu-\kappa} & 0 \\ 0 & \mathbb{L}_\kappa^{\nu+\kappa} \end{pmatrix} - \begin{pmatrix} \nu-\kappa & 0 \\ 0 & -(\nu+\kappa) \end{pmatrix}\right\}\mathbf{S}_\kappa. \tag{16}$$

Proof. By considering the unitary operator of multiplication by $(1+\kappa|z|^2)$ from $L^2(M_\kappa; \mathrm{d}m)$ onto $L^2(M_\kappa; \mathrm{d}\mu_\kappa)$, i.e.,

$$\begin{aligned} L^2(M_\kappa; \mathrm{d}m) &\longrightarrow L^2(M_\kappa; \mathrm{d}\mu_\kappa) \\ f &\longmapsto (1+\kappa|z|^2)f =: \tilde{f}, \end{aligned}$$

and using direct computation we see that for arbitrary $\alpha, \beta \in \mathbb{R}$, we have

$$\mathbb{L}_\kappa^{\alpha,\beta}(f) = (1+\kappa|z|^2)^{-1}\mathbb{L}_\kappa^{\frac{\alpha+\beta}{2}}[(1+\kappa|z|^2)f]. \tag{17}$$

Hence, result (16) holds as immediate consequence of theorem 1 and (17). □

Therefore, equation (15) becomes equivalent to the following system:

$$\begin{cases} \mathbb{L}_\kappa^{\nu_1}\tilde{f} = (\lambda + 4\nu_1)\tilde{f} \\ \\ \mathbb{L}_\kappa^{\nu_2}\tilde{g} = (\lambda - 4\nu_2)\tilde{g} \end{cases} ; \qquad \tilde{f}, \tilde{g} \in L^2(M_\kappa; \mathrm{d}\mu_\kappa), \tag{18}$$

with $\nu_1 = \nu - \kappa$ and $\nu_2 = \nu + \kappa$. Thus, using (i) of proposition 2, we get

Proposition 3. *The L^2-eigenvalues of the Pauli Hamiltonian $\mathbb{P}_\kappa^\nu$ acting on $\mathcal{H}_\kappa^+ = L^2(M_\kappa; \mathrm{d}m)\,\mathrm{d}z$ are given by*

$$\lambda_\kappa^+(l) = 2\nu l + \kappa l(l-1), \qquad l = 0, 1, 2, \ldots, \tag{19}$$

with $0 \leqslant l < \nu\rho^2 + \frac{1}{2}$ for the disc D_ρ and $2\nu\rho^2 = 2, 3, \ldots,$ for S_ρ^2. Whereas, the L^2-eigenvalues of $\mathbb{P}_\kappa^\nu$ acting on $\mathcal{H}_\kappa^- = L^2(M_\kappa; \mathrm{d}m)\,\mathrm{d}\bar{z}$ are given by

$$\lambda_\kappa^-(l') = 2\nu(l'+1) + \kappa(l'+1)(l'+2), \qquad l' = 0, 1, 2, \ldots, \tag{20}$$

with $0 \leqslant l' < \nu\rho^2 - \frac{3}{2}$ for D_ρ and $2\nu\rho^2 = 1, 2, \ldots,$ for S_ρ^2.

Furthermore, if ∇_α is the first-order differential operator defined by

$$\nabla_\alpha := (1+\kappa|z|^2)\frac{\partial}{\partial z} + (\alpha+\kappa)\bar{z}, \qquad \alpha \in \mathbb{R},$$

then we have

Theorem 2. *The differential form $f\,dz$ (resp. $g\,d\bar{z}$) is the solution of $\mathbb{P}_\kappa^\nu\omega = \lambda\omega$ in $\mathcal{H}_\kappa^+$ (resp. in $\mathcal{H}_\kappa^-$) if and only if $\lambda = \lambda_\kappa^+(l)$ (resp. $\lambda = \lambda_\kappa^-(l')$) and f (resp. g) can be expanded in $L^2(\mathbb{C}; dm)$ as follows*

$$f(z) = (1+\kappa|z|^2)^{-1}\nabla_{\nu-\kappa}\circ\nabla_\nu\circ\cdots\circ\nabla_{\nu+(l-2)\kappa}[(1+\kappa|z|^2)^{-\frac{\nu}{\kappa}-l+1}h(z)]$$

$$(\text{resp. } g(z) = (1+\kappa|z|^2)^{-1}\nabla_{\nu+\kappa}\circ\nabla_{\nu+2\kappa}\circ\cdots\circ\nabla_{\nu+l'\kappa}[(1+\kappa|z|^2)^{-\frac{\nu}{\kappa}-l'-1}h(z)]),$$

where $h(z) = \sum_{p=0}^{+\infty} a_p z^p$ is an arbitrary holomorphic function on M_κ whose coefficients a_p satisfy the growth condition given in (13) for $\tilde{\nu} = \nu - \kappa$ (resp. $\tilde{\nu} = \nu + \kappa$).

Proof. Let $f\,dz \in \mathcal{H}_\kappa^+$ be a solution of $\mathbb{P}_\kappa^\nu(f\,dz) = \lambda_\kappa^+(l)f\,dz$. Then, the function $\tilde{f} = (1+\kappa|z|^2)f \in L^2(M_\kappa; d\mu_\kappa)$ is a L^2-eigenfunction of the Landau Hamiltonian $\mathbb{L}_\kappa^{\tilde{\nu}}$ ($\tilde{\nu} = \nu - \kappa$) with $\mu_\kappa(l)$ as eigenvalue. Hence, $\tilde{f}$ can be expanded as in (11)–(13).

To conclude for theorem 2, it suffices to see that every component $\phi_{\kappa,l}^{\nu,pq}$ can be written in terms of the first-order differential operator ∇_α. We claim

Lemma 3. *Set $\nabla_{\tilde{\nu},m} := \nabla_{\tilde{\nu}}\circ\nabla_{\tilde{\nu}+\kappa}\circ\cdots\circ\nabla_{\tilde{\nu}+(m-1)\kappa}$. Then, in terms of the Gauss hypergeometric function ${}_2F_1$, we have*

$$\nabla_{\tilde{\nu},m}\left[(1+\kappa|z|^2)^{-\frac{\tilde{\nu}}{\kappa}}z^p\right] = C_\kappa^{\tilde{\nu}}(p,m)(1+\kappa|z|^2)^{-\frac{\tilde{\nu}}{\kappa}-m}|z|^{|m-p|}\,e^{-i(m-p)\arg z}$$
$$\times\, {}_2F_1\left(-Min(p,m), Max(p-m,0) - m - \frac{2\tilde{\nu}}{\kappa}; |m-p|+1; -\kappa|z|^2\right)$$

Whose proof can be handled by induction using some known transformations on hypergeometric functions. □

Remark 3. Here we have given the result in a unified manner for both positive and negative constant curvature κ, which recovers also the flat case ($\kappa = 0$). In the particular case of the Pauli Hamiltonian on the sphere S_ρ^2 ($\kappa = +1/\rho^2$), the result of theorem 2 can also be deduced from theorem 3 of [6].

4. Explicit formulae for the L^2-eigenprojector kernels of $\mathbb{P}_\kappa^\nu$

Fix $\nu > 0$ and let $A_{l,l'}^{2,\nu}(\mathbb{P}_\kappa^\nu)$ be the L^2-eigenspace of the Pauli Hamiltonian $\mathbb{P}_\kappa^\nu$ associated with the L^2-eigenvalue $\lambda = \lambda_\kappa(l,l')$. That is,

$$A_{l,l'}^{2,\nu}(\mathbb{P}_\kappa^\nu) = \left\{\omega = \begin{pmatrix} f \\ g \end{pmatrix} \in L^2(M_\kappa; dm)\oplus L^2(M_\kappa; dm); \mathbb{P}_\kappa^\nu\omega = \lambda_\kappa(l,l')\omega\right\}.$$

Then, the Hilbert space $A_{l,l'}^{2,\nu}(\mathbb{P}_\kappa^\nu)$ admits a L^2-eigenprojector kernel (reproducing kernel) $\mathcal{K}_{\kappa;l,l'}^\nu(z,w)$, i.e., such that for all $\omega \in A_{l,l'}^{2,\nu}(\mathbb{P}_\kappa^\nu)$ we have

$$\omega(z) = \int_{M_\kappa}\mathcal{K}_{\kappa;l,l'}^\nu(z,w)\omega(w)\,d\mu_\kappa(w).$$

Moreover, it is given explicitly by the following:

Theorem 3. *Fix $l, l' \in \mathbb{Z}^+$ such that $\lambda_\kappa^+(l) = \lambda_\kappa^-(l') = \lambda_\kappa(l,l')$. Then, the L^2-eigenprojector kernel $\mathcal{K}_{\kappa;l,l'}^\nu(z,w)$ of the L^2-eigenspace $A_{l,l'}^{2,\nu}(\mathbb{P}_\kappa^\nu)$ is given by*

$$\mathcal{K}_{\kappa;l,l'}^\nu(z,w) = \left(\frac{1+\kappa|w|^2}{1+\kappa|z|^2}\right)\begin{pmatrix}\tilde{\mathcal{K}}_{\kappa,l}^{\nu-\kappa}(z,w) & 0 \\ 0 & \tilde{\mathcal{K}}_{\kappa,l'}^{\nu+\kappa}(z,w)\end{pmatrix},$$

where $\tilde{\mathcal{K}}^{\tilde{\nu}}_{\kappa;m}(z,w)$ *is given explicitly by the following closed formula:*

$$\tilde{\mathcal{K}}^{\tilde{\nu}}_{\kappa;m}(z,w) = \frac{2\tilde{\nu}+\kappa(2m+1)}{\pi}\left(\frac{1+\kappa\bar{z}w}{1+\kappa z\bar{w}}\right)^{-\frac{\tilde{\nu}}{\kappa}} \times \left(\frac{|1+\kappa z\bar{w}|^2}{(1+\kappa|z|^2)(1+\kappa|w|^2)}\right)^{\frac{\tilde{\nu}}{\kappa}+m} {}_2F_1\left(-m, -\frac{2\tilde{\nu}}{\kappa}-m; 1; -\kappa\frac{|z-w|^2}{|1+\kappa z\bar{w}|^2}\right). \tag{21}$$

Proof. In view of (18), we split the Hilbert space $A^{2,\nu}_{l,l'}(\mathbb{P}^\nu_\kappa)$ in a direct sum as follows:

$$A^{2,\nu}_{l,l'}(\mathbb{P}^\nu_\kappa) = (1+\kappa|z|^2)^{-1}A^{2,\nu}_{\mu^+_\kappa(l)}(\mathbb{L}^{\nu-\kappa}_\kappa)\,\mathrm{d}z \oplus (1+\kappa|z|^2)^{-1}A^{2,\nu}_{\mu^-_\kappa(l')}(\mathbb{L}^{\nu+\kappa}_\kappa)\,\mathrm{d}\bar{z},$$

where $A^{2,\nu}_{\mu^\bullet_\kappa(m)}(\mathbb{L}^{\tilde{\nu}}_\kappa)(\subset L^2(M_\kappa;\mathrm{d}\mu_\kappa))$, for $\tilde{\nu}=\nu-\kappa$ and $m=l$ or $\tilde{\nu}=\nu+\kappa$ and $m=l'$, is the L^2-eigenspace of the Landau Hamiltonian $\mathbb{L}^{\tilde{\nu}}_\kappa$ associated with the Landau level $\mu^\bullet_\kappa(m)$ (with $\mu^+_\kappa(l)=\lambda^+_\kappa(l)+(\nu-\kappa)$ and $\mu^-_\kappa(l')=\lambda^-_\kappa(l')-(\nu+\kappa)$). Therefore, $A^{2,\nu}_{l,l'}(\mathbb{P}^\nu_\kappa)$ admits a L^2-eigenprojector kernel $\mathcal{K}^\nu_{\kappa;l,l'}(z,w)$ which is equal to

$$\mathcal{K}^\nu_{\kappa;l,l'}(z,w) = \frac{1}{\sqrt{2}}(1+\kappa|z|^2)^{-1}\begin{pmatrix}\tilde{\mathcal{K}}^{\nu-\kappa}_{\kappa;l}(z,w) & 0\\ 0 & \tilde{\mathcal{K}}^{\nu+\kappa}_{\kappa;l'}(z,w)\end{pmatrix}(1+\kappa|w|^2),$$

where $\tilde{\mathcal{K}}^{\tilde{\nu}}_{\kappa,m}(z,w)$ is the L^2-eigenprojector kernel of the L^2-eigenspace $\widetilde{A}^{2,\tilde{\nu}}_{\mu^\bullet_\kappa(m)}$ of $\mathbb{L}^{\tilde{\nu}}_\kappa$ associated with the Landau level $\tilde{\nu}(2m+1)+\kappa m(m+1)$. Then, we need just to specify further $\tilde{\mathcal{K}}^{\tilde{\nu}}_{\kappa,m}(z,w)$, which is given by the following well-established proposition.

Proposition 4. *The* L^2*-eigenprojector kernel* $\tilde{\mathcal{K}}^{\tilde{\nu}}_{\kappa,m}(z,w)$ *of the Landau Hamiltonian* $\mathbb{L}^{\tilde{\nu}}_\kappa$ *associated to the Landau level* $\tilde{\nu}(2m+1)+\kappa m(m+1)$ *is given explicitly by*

$$\tilde{\mathcal{K}}^{\tilde{\nu}}_{\kappa,m}(z,w) = \frac{2\tilde{\nu}+\kappa(2m+1)}{\pi}\left(\frac{1+\kappa\bar{z}w}{1+\kappa z\bar{w}}\right)^{-\frac{\tilde{\nu}}{\kappa}} \times \left(\frac{|1+\kappa z\bar{w}|^2}{(1+\kappa|z|^2)(1+\kappa|w|^2)}\right)^{\frac{\tilde{\nu}}{\kappa}+m} {}_2F_1\left(-m, -\frac{2\tilde{\nu}}{\kappa}-m; 1; -\kappa\frac{|z-w|^2}{|1+\kappa z\bar{w}|^2}\right). \tag{22}$$

Thus, the proof of theorem 3 will be completed by proving (22) (for its proof see appendix B). □

Remark 4. If $\lambda^+_\kappa(l)\neq\lambda^-_\kappa(l')$, we have to consider two cases. The case $\lambda_\kappa(l,l')=\lambda^+_\kappa(l)\neq\lambda^-_\kappa(l')$, for which we have

$$\mathcal{K}^\nu_{\kappa;l}(z,w) = \left(\frac{1+\kappa|w|^2}{1+\kappa|z|^2}\right)\begin{pmatrix}\tilde{\mathcal{K}}^{\nu-\kappa}_{\kappa;l}(z,w) & 0\\ 0 & 0\end{pmatrix}$$

and the case $\lambda_\kappa(l,l')=\lambda^-_\kappa(l')\neq\lambda^+_\kappa(l)$, for which we have

$$\mathcal{K}^\nu_{\kappa,l'}(z,w) = \left(\frac{1+\kappa|w|^2}{1+\kappa|z|^2}\right)\begin{pmatrix}0 & 0\\ 0 & \tilde{\mathcal{K}}^{\nu+\kappa}_{\kappa,l'}(z,w)\end{pmatrix}.$$

Particularly, for the bottom eigenvalue 0 (corresponding to $\lambda^+_\kappa(0)=0$ for $l=0$), we have

$$A^{2,\nu}_0(\mathbb{P}^\nu_\kappa) = \left\{\omega=\begin{pmatrix}f\\0\end{pmatrix};\ f\in L^2(M_\kappa;\mathrm{d}m);\ \mathbb{P}^\nu_\kappa(f\,\mathrm{d}z)=0\right\},$$

and the associated kernel function $\mathcal{K}^{\nu}_{\kappa,0}(z,w)$ is given by

$$\mathcal{K}^{\nu}_{\kappa,0}(z,w) = \frac{2\nu-\kappa}{\pi}\left(\frac{1+\kappa|w|^2}{1+\kappa|z|^2}\right)\left(\frac{|1+\kappa z\bar{w}|^2}{(1+\kappa|z|^2)(1+\kappa|w|^2)}\right)^{\frac{\nu}{\kappa}-1}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

5. Euclidean limit

According to the above discussion, the following facts hold when $\kappa \to 0$ (i.e., when the radius ρ goes to $+\infty$).

Fact 1. The group of motions G_κ on M_κ converges to the affine group on the Euclidean plane $\mathbb{R}^2 = \mathbb{C}$. More exactly, viewing the groups of motions $G_\kappa, \kappa < 0, \kappa = 0, \kappa > 0$, as subspaces of $M_{2,2}(\mathbb{C})$, then for every fixed g_0 in the affine group $G_0 = \left\{\begin{pmatrix} a & b\\ 0 & \bar{a}\end{pmatrix} \in M_{2,2}(\mathbb{C}); |a|^2 = 1, b \in \mathbb{C}\right\}$, there exists a family $(g_\kappa)_\kappa$ depending smoothly on κ such that $g_\kappa = \begin{pmatrix} a(\kappa) & b(\kappa)\\ -\kappa\bar{b}(\kappa) & \bar{a}(\kappa)\end{pmatrix} \in G_\kappa$ for every κ and $g_0 = \lim_{\kappa\to 0} g_\kappa$. Conversely, let $(g_\kappa)_\kappa$ be such that $g_\kappa \in G_\kappa$ for every κ and $\lim_{\kappa\to 0} g_\kappa$ exists. Then, we have $\lim_{\kappa\to 0} g_\kappa \in G_0$. Further details can be found in [7] for a more general setting.

Fact 2. From (19) and (20), it is clear that the L^2-eigenvalues of the Pauli Hamiltonian $\mathbb{P}^{\nu}_{\kappa}$ on M_κ converge to $2\nu l, l = 0, 1, \ldots,$ that constitute the point spectrum of $\mathbb{P}^{\nu}$, the Pauli Hamiltonian on the plane $\mathbb{R}^2 = \mathbb{C}$. This is exactly Comtet's result [3] for the Landau Hamiltonian with $\kappa < 0$.

Fact 3. The L^2-eigenfunctions $\phi^{\tilde{\nu},pq}_{\kappa,m}(z)$ given in (12) associated with the L^2-eigenvalue $\mu_\kappa(m) = \tilde{\nu}(2m+1) + \kappa m(m+1)$ and realized in lemma 3, up to a given multiplicative constant, as follows:

$$\nabla_{\tilde{\nu}} \circ \nabla_{\tilde{\nu}+\kappa} \circ \cdots \circ \nabla_{\tilde{\nu}+(m-1)\kappa}\left[(1+\kappa|z|^2)^{-\frac{\tilde{\nu}}{\kappa}} z^j\right]$$

gives rise, when κ tends to 0 and for every fixed $z \in \mathbb{C}$, to

$$\lim_{\kappa\mapsto 0} \phi^{\tilde{\nu},pq}_{\kappa,m}(z) = \left(\frac{\partial}{\partial z} + \nu\bar{z}\right)^m \left(\mathrm{e}^{-\nu|z|^2} z^j\right),$$

which constitute an orthogonal basis of the L^2-eigenspace of the Landau Hamiltonian $\mathbb{L}^{\nu}$ for the flat case corresponding to the Landau level $\nu(2m+1) = \lim_{\kappa\to 0}\mu_\kappa(m)$. They are expressed in terms of the confluent hypergeometric function ${}_1F_1(a; c; x)$, up to a given multiplicative constant, as follows:

$$\mathrm{e}^{-\nu|z|^2}|z|^{|p-m|}\,\mathrm{e}^{-\mathrm{i}(m-p)\arg z}\,{}_1F_1(-Min(m,p); |p-m|+1; 2\nu|z|^2).$$

Now, for the asymptotic of the L^2-eigenprojector kernels of $\mathbb{P}^{\nu}_{\kappa}$, we have the following main result.

Theorem 4. *Fix $\nu > 0$ and let $\kappa = \varepsilon/\rho^2$ for $\varepsilon = \mp 1$ and ρ varying such that $\rho^2 \in (1/2\nu)\mathbb{Z}^+$. Then, in the limit $\kappa \to 0$, the L^2-eigenprojector kernel $\mathcal{K}^{\nu}_{\kappa;l,l'}(z,w)$ of $\mathbb{P}^{\nu}_{\kappa}$ converges pointwisely on $\mathbb{C}\times\mathbb{C}$ to the L^2-eigenprojector kernel of $\mathbb{P}^{\nu}$ on $\mathbb{C}$.*

Proof. Let us note first that for κ small enough the condition $2\tilde{\nu} + \kappa > 0$ in proposition 2 for $\kappa = -1/\rho^2$ is always satisfied and the assumption $\lambda^{+}_{\kappa}(l) = \lambda^{-}_{\kappa}(l')$ in theorem 3 reads simply $l = l'+1$, so that the L^2-eigenprojector kernel $\mathcal{K}^{\nu}_{\kappa;l,l'}(z,w)$ of the Pauli Hamiltonian $\mathbb{P}^{\nu}_{\kappa}$ is well defined for a fixed $(z, w) \in \mathbb{C} \times \mathbb{C}$. Then, keeping in mind the explicit formula of $\mathcal{K}^{\nu}_{\kappa;l,l'}(z, w)$ (see theorem 3), we get

$$\lim_{\kappa \mapsto 0} \frac{(2\tilde{\nu} + (2m+1)\kappa)}{\pi} \left(\frac{1 + \kappa \bar{z} w}{1 + \kappa z \bar{w}} \right)^{-\frac{\tilde{\nu}}{\kappa}} = \frac{2\nu}{\pi} \, \mathrm{e}^{\nu(z\bar{w} - \bar{z}w)}$$

and

$$\lim_{\kappa \mapsto 0} \left(\frac{|1 + \kappa z \bar{w}|^2}{(1 + \kappa |z|^2)(1 + \kappa |w|^2)} \right)^{\frac{\tilde{\nu}}{\kappa} + m} = \mathrm{e}^{-\nu |z - w|^2}$$

for $\tilde{\nu} = \nu - \kappa$ or $\nu + \kappa$. Then, using the well-known fact

$$\lim_{t \mapsto 0} {}_2F_1\left(a, b + t; c; \frac{x}{t} \right) = {}_1F_1(a; c; x),$$

we conclude that

$$\lim_{\kappa \mapsto 0} {}_2F_1 \left(-l, -\frac{2\tilde{\nu}}{\kappa} - l; 1; -\kappa \frac{|z - w|^2}{|1 + \kappa z \bar{w}|^2} \right) = {}_1F_1(-l; 1; 2\nu |z - w|^2)$$

and

$$\lim_{\kappa \mapsto 0} {}_2F_1 \left(-l', -\frac{2\tilde{\nu}}{\kappa} - l'; 1; -\kappa \frac{|z - w|^2}{|1 + \kappa z \bar{w}|^2} \right) = {}_1F_1(1 - l; 1; 2\nu |z - w|^2)$$

for $l = l' + 1$. Therefore, we have

$$\lim_{\kappa \mapsto 0} \mathcal{K}^{\nu}_{\kappa;l,l'}(z, w) = \frac{2\nu}{\pi} \, \mathrm{e}^{\nu(z\bar{w} - \bar{z}w)} \, \mathrm{e}^{-\nu|z-w|^2} \times \begin{pmatrix} {}_1F_1(-l; 1; 2\nu|z - w|^2) & 0 \\ 0 & {}_1F_1(1 - l; 1; 2\nu|z - w|^2) \end{pmatrix}.$$

Finally, note that the right-hand side is nothing but the L^2-eigenprojector kernel $\mathcal{K}^{\nu}_{0,l}(z, w)$ of the Pauli Hamiltonian $\mathbb{P}^{\nu}$ of the Euclidean plane $\mathbb{C}$. In fact, we have

Lemma 4. *For $l = 0$, the L^2-eigenprojector kernel of the L^2-eigenspace $A^{2,\nu}_0(\mathbb{P}^{\nu})$ is given by*

$$\frac{2\nu}{\pi} \, \mathrm{e}^{\nu(z\bar{w} - \bar{z}w)} \, \mathrm{e}^{-\nu|z-w|^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $l \neq 0$, the L^2-eigenprojector kernel of the L^2-eigenspace $A^{2,\nu}_l(\mathbb{P}^{\nu})$ is given by

$$\mathcal{K}^{\nu}_{0,l}(z, w) = \frac{2\nu}{\pi} \, \mathrm{e}^{\nu(z\bar{w} - \bar{z}w)} \, \mathrm{e}^{-\nu|z-w|^2} \begin{pmatrix} {}_1F_1(-l; 1; 2\nu|z - w|^2) & 0 \\ 0 & {}_1F_1(1 - l; 1; 2\nu|z - w|^2) \end{pmatrix}.$$

This finishes the proof of theorem 4. □

6. Concluding remarks

In this paper, we have provided concrete spectral properties, such as the explicit formulae for the L^2-eigenforms and the L^2-eigenprojector kernels, of the Pauli Hamiltonians $\mathbb{P}^{\nu}_{\kappa}$ on simply connected Riemann surfaces with constant scalar curvature $\kappa = \varepsilon/\rho^2$. Further, we have discussed their limits when $\kappa \to 0$ (i.e. $\rho \to +\infty$). This was possible thanks to the explicit expression of $\mathbb{P}^{\nu}_{\kappa}$ (theorem 1). Below, we point out some related remarks.

In contrast to the case of the scalar Landau Hamiltonians $\mathbb{L}^{\nu}_{\kappa}$, the value 0 is an isolated L^2-eigenvalue for the Pauli Hamiltonians $\mathbb{P}^{\nu}_{\kappa}$. The corresponding L^2-eigenspace reduces further to $\ker \mathbb{P}_\kappa^\nu$ in $\mathcal{H}_\kappa$. More exactly, for the hyperbolic disc D_ρ, the corresponding L^2-eigenspace $\ker \mathbb{P}_\kappa^\nu$ given by

$$\left\{\begin{pmatrix} f \\ 0 \end{pmatrix}; f(z) = \left(1 - \left|\frac{z}{\rho}\right|^2\right)^{\nu\rho^2} \sum_{p=0}^{+\infty} a_p z^p \text{ with } \sum_{p=0}^{+\infty} \frac{\rho^{2p} p! \Gamma(2\nu\rho^2 + 2)|a_p|^2}{\Gamma(2\nu\rho^2 + p + 2)} < +\infty\right\},$$

is isomorphic to the usual weighted Bergman Hilbert space of square integrable holomorphic functions on D_ρ with respect to the density $\left(1 - \left|\frac{z}{\rho}\right|^2\right)^{2\nu\rho^2-2} \mathrm{d}m$. For S_ρ^2, the null space of the Pauli Hamiltonian is reduced essentially, up to the multiplicative function $\left(1 + \left|\frac{z}{\rho}\right|^2\right)^{-\nu\rho^2}$, to the space of polynomial functions of degrees less than or equal to the integer $2\nu\rho^2$. Therefore, it results that the formal limit of such Hilbert spaces, when $\rho \to +\infty$, gives rise to $\ker \mathbb{P}^\nu$ which is isomorphic to the classical Bargmann–Fock space

$$\left\{h : \mathbb{C} \to \mathbb{C}, \quad h \text{ entire on } \mathbb{C} \text{ and } \int_{\mathbb{C}} |h(z)|^2 \, \mathrm{e}^{-2\nu|z|^2} \, \mathrm{d}m(z) < +\infty\right\}.$$

Also, we note that the L^2-eigenvalues $\lambda_\kappa^+(l)$ and $\lambda_\kappa^-(l')$ of the Pauli Hamiltonian $\mathbb{P}_\kappa^\nu$ occur with infinite multiplicities for $\kappa \leqslant 0$. For $\kappa = +1/\rho^2 > 0$, they occur with finite degeneracies. More precisely, for the eigenvalue $2\nu l + \kappa l(l-1)$, the multiplicity is $2\nu\rho^2 + 2l - 1$ and for the eigenvalue $2\nu(l+1) + \kappa(l+1)(l+2)$, the multiplicity is $2\nu\rho^2 + 2l + 3$. Therefore, when the planar image is taken we recuperate the eigenvalues with their multiplicities.

We conclude this section by noting that under the assumptions $2\nu\rho^2 > 3$ for the disc D_ρ and $2\nu\rho^2 = 2, 3, \ldots,$ for S_ρ^2, the existence of a L^2-eigenform $\omega = f\,\mathrm{d}z + g\,\mathrm{d}\bar{z}$ with $f \neq 0$ and $g \neq 0$ is related to the existence of a solution (l, l') of the following algebraic equation:

$$l(2\nu\rho^2 + 1 - l) = (l' + 1)(2\nu\rho^2 - 2 - l'). \tag{23}$$

It depends clearly on the value of $2\nu\rho^2$. Thus, if $2\nu\rho^2$ is irrational there is no solution of (23). Further, one can show that (l, l') to be a solution for (23), we must have necessary $l' \geqslant l$. So, $(l, l+s)$ for $s \in \mathbb{Z}^+$, is a solution if and only if $2\nu\rho^2$ is rational such that $(2l + s + 1)(s + 2) = 2\nu\rho^2(s + 1)$. Note finally that such equation (23) reads simply $l = l' + 1$ for ρ large enough.

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Appendix A. Proof of theorem 1

We first fix further notation and recall some needed facts. For $p = 0, 1$ or 2, let $\mathcal{H}_\kappa^p$ denote the Hilbert space obtained as the completion of $\mathcal{C}_c^\infty(M_\kappa; \Lambda^p M_\kappa)$, the space of $\mathcal{C}^\infty$-differential p-forms with compact support in M_κ, with respect to the natural Hermitian scalar product induced from the metric $\mathrm{d}s_\kappa^2$ and defined by

$$(\alpha, \beta)_p := \int_{M_\kappa} \alpha \wedge \star\beta, \qquad \alpha, \beta \in \mathcal{C}_c^\infty(M_\kappa; \Lambda^p M_\kappa). \tag{A.1}$$

In the integrand, $\star$ is the Hodge star operator canonically associated with $\mathrm{d}s_\kappa^2$ acting on differential forms of M_κ as follows

$$\star 1 = 2\mathrm{i}\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+\kappa|z|^2)^2}; \quad \star\mathrm{d}z = \mathrm{i}\,\mathrm{d}\bar{z}; \quad \star\mathrm{d}\bar{z} = -\mathrm{i}\,\mathrm{d}z; \quad \star(\mathrm{d}z\wedge\mathrm{d}\bar{z}) = \frac{\mathrm{i}}{2}(1+\kappa|z|^2)^2.$$

At this level let mention that the adjoints d^* of d and $(\mathbf{ext}\,\theta_\kappa)^*$ of $\mathbf{ext}\,\theta_\kappa$, with respect to the Hermitian scalar product (A.1), are given respectively by

$$d^* = -\star d \star \qquad \text{and} \qquad (\mathbf{ext}\,\theta_\kappa)^* = \star(\mathbf{ext}\,\theta_\kappa)\star.$$

Then, for $p = 0$, the space $\mathcal{H}_\kappa^0$ is nothing but the usual Hilbert space of square integrable complex-valued functions on M_κ with respect to the volume measure $\mathrm{d}\mu_\kappa$, i.e., $\mathcal{H}_\kappa^0 = L^2(M_\kappa; \mathrm{d}\mu_\kappa)$. For $p = 1$, the Hilbert space $\mathcal{H}_\kappa^1$ of L^2-differential 1-forms $\omega = f\,\mathrm{d}z + g\,\mathrm{d}\bar{z}$ consists of the couple of functions (f, g) on M_κ that are square integrables with respect to the Lebesgue measure $\mathrm{d}m$. That is

$$\mathcal{H}_\kappa^1 = L^2(M_\kappa; \mathrm{d}m)\,\mathrm{d}z \oplus L^2(M_\kappa; \mathrm{d}m)\,\mathrm{d}\bar{z} =: \mathcal{H}_\kappa.$$

Now, let us proceed to the proof of theorem 1.

Proof. Note first that the Pauli Hamiltonian $\mathbb{P}_\kappa^\nu$,

$$\mathbb{P}_\kappa^\nu = (d + \mathrm{i}\nu\,\mathbf{ext}\,\theta_\kappa)^*(d + \mathrm{i}\nu\,\mathbf{ext}\,\theta_\kappa) + (d + \mathrm{i}\nu\,\mathbf{ext}\,\theta_\kappa)(d + \mathrm{i}\nu\,\mathbf{ext}\,\theta_\kappa)^*,$$

can be rewritten in the following form:

$$\mathbb{P}_\kappa^\nu = [d, d^*]_+ + \mathrm{i}\nu([d^*, \mathbf{ext}\,\theta_\kappa]_+ - [d, (\mathbf{ext}\,\theta_\kappa)^*]_+) + \nu^2[\mathbf{ext}\,\theta_\kappa, (\mathbf{ext}\,\theta_\kappa)^*]_+,$$

where $[A, B]_+ = AB + BA$. Therefore, for $\omega = f\,\mathrm{d}z + g\,\mathrm{d}\bar{z} \equiv \binom{f}{g}$ being an arbitrary smooth differential 1-form on M_κ and $h_\kappa(z) := 1 + \kappa|z|^2$, we get the following results by direct computation:

Lemma 5. *The Hodge–de Rham operator $d^*d + dd^*$ acting on ω is given by the following explicit expression in the complex coordinate z:*

$$d^*d + dd^* = -h_\kappa(z)\left\{h_\kappa(z)\begin{pmatrix}\partial^2/\partial z\partial\bar{z} & 0\\ 0 & \partial^2/\partial z\partial\bar{z}\end{pmatrix} + 2\kappa\begin{pmatrix}\overline{E} & 0\\ 0 & E\end{pmatrix}\right\},$$

where $\overline{E}$ is the complex conjugate of the complex Euler operator $E = z\partial/\partial z$.

Lemma 6. *The operators $[\mathbf{ext}\,\theta_\kappa, (\mathbf{ext}\,\theta_\kappa)^*]_+$, $[d, (\mathbf{ext}\,\theta_\kappa)^*]_+$ and $[d^*, \mathbf{ext}\,\theta_\kappa]_+$ act by*

(i) $[\mathbf{ext}\,\theta_\kappa, (\mathbf{ext}\,\theta_\kappa)^*]_+\omega = |z|^2\omega$.
(ii) $[d^*, \mathbf{ext}\,\theta_\kappa]_+\omega = \frac{\mathrm{i}}{2}[h_\kappa(z)(E - \bar{E})\omega + h_\kappa(z)\sigma_3\omega + \kappa T_z\omega]$.
(iii) $[d, (\mathbf{ext}\,\theta_\kappa)^*]_+\omega = -\frac{\mathrm{i}}{2}[h_\kappa(z)(E - \bar{E})\omega + h_\kappa(z)\sigma_3\omega - \kappa T_z\omega]$,

where $\sigma_3 := \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $T_z := \begin{pmatrix}0 & \bar{z}^2\\ -z^2 & 0\end{pmatrix}$, i.e., $\sigma_3(f\,\mathrm{d}z + g\,\mathrm{d}\bar{z}) := f\,\mathrm{d}z - g\,\mathrm{d}\bar{z}$ and $T_z(f\,\mathrm{d}z + g\,\mathrm{d}\bar{z}) = \bar{z}^2 g\,\mathrm{d}z - z^2 f\,\mathrm{d}\bar{z}$.

Thus, by combining the above results, we obtain the desired explicit expression of $\mathbb{P}_\kappa^\nu$ of theorem 1.

Appendix B. Sketch of the proof of proposition 4

The proof of this proposition can be checked in a similar way as in [11] or [7].

First note that using the invariance property of the Landau Hamiltonian $\mathbb{L}_\kappa^{\tilde{\nu}}$ by the group of motions G_κ (8), it follows that the L^2-eigenprojector kernel $\tilde{\mathcal{K}}_{\kappa,m}^{\tilde{\nu}}(z, w)$ satisfies the following invariance property:

Lemma 7. *For all $g \in G_\kappa$, we have*

$$\tilde{\mathcal{K}}_{\kappa,m}^{\tilde{\nu}}(z, w) = \left(\frac{1+\kappa z\overline{g^{-1}.0}}{1+\kappa\bar{z}g^{-1}\cdot 0}\right)^{\frac{\tilde{\nu}}{\kappa}} \left(\frac{1+\kappa w\overline{g^{-1}\cdot 0}}{1+\kappa\bar{w}g^{-1}\cdot 0}\right)^{-\frac{\tilde{\nu}}{\kappa}} \tilde{\mathcal{K}}_{\kappa,m}^{\tilde{\nu}}(g\cdot z, g\cdot w). \tag{B.1}$$

Therefore, for $g = g_w := (1+\kappa|w|^2)^{-\frac{1}{2}}\left(\begin{smallmatrix} 1 & w \\ -\kappa\bar{w} & 1\end{smallmatrix}\right)$, which is clearly in G_κ and satisfies $g_w\cdot w = 0$ and $g_w^{-1}\cdot 0 = w$, we conclude (from (B.1)) that

$$\tilde{\mathcal{K}}_{\kappa,m}^{\tilde{\nu}}(z, w) = \left(\frac{1+\kappa\bar{z}w}{1+\kappa z\bar{w}}\right)^{-\frac{\tilde{\nu}}{\kappa}} \tilde{\mathcal{K}}_{\kappa,m}^{\tilde{\nu}}\big(g_w^{-1}\cdot z, 0\big). \tag{B.2}$$

In addition, we have

Lemma 8. *The function $Z \in M_\kappa \mapsto \tilde{\mathcal{K}}_{\kappa,m}^{\tilde{\nu}}(Z, 0)$ is a radial L^2-eigenfunction of $\mathbb{L}_\kappa^{\tilde{\nu}}$ with $\tilde{\nu}(2m+1)+\kappa m(m+1)$ as eigenvalue. More explicitly*

$$\tilde{\mathcal{K}}_{\kappa,m}^{\tilde{\nu}}(Z, 0) = \frac{2\tilde{\nu}+\kappa(2m+1)}{\pi}(1+\kappa|z|^2)^{-\frac{\tilde{\nu}}{\kappa}-m}{}_2F_1\left(-m, -\frac{2\tilde{\nu}}{\kappa}-m; 1; -\kappa|Z|^2\right). \tag{B.3}$$

Next, using the facts

$$\left|g_w^{-1}\cdot z\right|^2 = \frac{|z-w|^2}{|1+\kappa z\bar{w}|^2} \qquad \text{and} \qquad 1+\kappa\left|g_w^{-1}\cdot z\right|^2 = \frac{(1+\kappa|z|^2)(1+\kappa|w|^2)}{|1+\kappa z\bar{w}|^2}$$

combined with (B.2) and (B.3), we obtain the desired result of proposition 4. □

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